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Positive definite solutions of the matrix equations $X \pm A^* X^{-q} A = Q$ ☆

Vejdi I. Hasanov

Laboratory of Mathematical Modelling, Shumen University, Shumen 9712, Bulgaria

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Abstract

In this paper we investigate nonlinear matrix equations $X + A^* X^{-q} A = Q$ and $X - A^* X^{-q} A = Q$ where $q \in (0, 1]$. We derive necessary conditions and sufficient conditions for the existence of positive definite solutions for these equations. We provide a sufficient condition for the equation $X + A^* X^{-q} A = Q$ to have two different positive definite solutions and a sufficient condition for the equation $X - A^* X^{-q} A = Q$ to have a unique positive definite solution. We also propose iterative methods for obtaining positive definite solutions for these equations.

AMS classification: 15A24; 65H05

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1. Introduction

In this work we investigate the nonlinear matrix equations

$$X + A^* X^{-q} A = Q, \quad X - A^* X^{-q} A = Q, \tag{1}$$

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E-mail address: v.hasanov@fmi.shu-bg.net

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where $A \in \mathbb{C}^{m\times m}$ ($\mathbb{C}^{m\times m}$—set of all $m \times m$ complex matrices), Q is positive definite, X is an unknown matrix and $q \in (0, 1]$.

The equations

$$X + A^*X^{-1}A = Q, \quad X - A^*X^{-1}A = Q, \tag{2}$$

which are representative of Eq. (1) for $q = 1$ have many applications in: control theory; dynamic programming; statistics; Kalman filtering and etc., see [7,19] and the references therein. Eq. (2) have been studied by some authors [1,7–9,11,14,17,19,20] and different iterative methods for computing the positive definite solutions with linear and quadratic rate of convergence are proposed.

The first attempts to solve Eq. (1) for $q \neq 1$ ($q = 1/2$ and $Q = I$) are made by El-Sayed [4]. Later, for the equation $X + A^*X^{-1/2}A = I$ in [12] and for the equation $X - A^*X^{-1/2}A = I$ in [15] we obtain better sufficient conditions for existence of the positive definite solutions, after modification of El-Sayed's methods. El-Sayed and Ramadan investigate the equation $X - A^*X^{-1/2^m}A = I$ [6]. In [13] the author investigates the equations (1) for $q = 1/n,\ n \in \mathbb{N}$. Other researches are made by Liu and Gao for the equations $X^s \pm A^{\mathrm{T}}X^{-t}A = I$ with $s, t \in \mathbb{N}$ [16], and by Du and Hou for the operator equation $X^m + A^*X^{-n}A = I$ with $m, n \in \mathbb{N}$ [3], where $\mathbb{N}$ is the set of natural numbers. El-Sayed and Ran [5], and Ran and Reurings [18] investigate the general matrix equation

$$X + A^*\mathscr{F}(X)A = Q, \tag{3}$$

where $\mathscr{F}$ is a map from the set of all positive semidefinite matrices into in $\mathbb{C}^{m\times m}$.

We write $B > 0$ ($B \geqslant 0$) if the matrix B is Hermitian positive definite (semidefinite). If $B - C$ is Hermitian positive definite (semidefinite), then we write $B > C$ ($B \geqslant C$).

The investigations of the above mentioned authors are for specific values of q. In this work we generalize the known results to arbitrary $q \in (0, 1]$ and we obtain some new properties of the solutions. In the next two section we make detailed analysis of Eq. (1). We derive a sufficient condition for the equation $X + A^*X^{-q}A = Q$ to have two different positive definite solutions and a sufficient condition for the equation $X - A^*X^{-q}A = Q$ to have a unique positive definite solution. We prove that if the equation $X + A^*X^{-q}A = Q$ has a positive definite solution, then it has a maximal solution $X_L > 0$, such that for any solution $X > 0$, $X_L \geqslant X$. We also propose iterative methods for obtaining positive definite solutions for these equations.

We use $\|A\|$ to denote l_2 induced operator norm of the matrix A, i.e. $\|A\| = \sigma_1(A)$, where $\sigma_1(A) \geqslant \sigma_2(A) \geqslant \cdots \geqslant \sigma_m(A) \geqslant 0$ are the singular values of A in nonincreasing order. By $\lambda_1(Q), \lambda_m(Q)$, we denote the maximal and the minimal eigenvalues of Q, respectively. Let $\mathscr{P}(m)$ denotes a set of $m \times m$ positive semidefinite matrices and

$$[A, B] = \{X|A \leqslant X \leqslant B\}, \quad (A, B] = \{X|A < X \leqslant B\}.$$

2. The equation $X + A^*X^{-q}A = Q$

In this section we concentrate on the equation

$$X + A^*X^{-q}A = Q. \tag{4}$$

It is obvious that Eq. (4) can be derived from Eq. (3) if $\mathscr{F}(X) = X^{-q}$.

We begin this section and the next section with some statements which are proved by Ran and Reurings in [18] for Eq. (3).

Theorem 2.1 [18, Lemma 2.1]. *Let $\mathscr{F} : \mathscr{P}(m) \to \mathscr{P}(m)$ be continuous on* $[0, Q]$.

(i) *If* (3) *has a positive semidefinite solution $\overline{X}$, then $\overline{X} \leqslant Q$ and $A^*\mathscr{F}(\overline{X})A \leqslant Q$.*

(ii) *If $A^*\mathscr{F}(X)A \leqslant Q$ for all $X \in [0, Q]$, then Eq.* (3) *has a solution in* $[0, Q]$.

Because $\mathscr{F}(X) = X^{-q}$ is not continuous on $[0, Q]$, Theorem 2.1 is not applicable to Eq. (4). In the next theorems, we give results for Eq. (4) similar to Theorem 2.1.

Theorem 2.2. *If Eq.* (4) *has a positive definite solution X, then*

$$\left(Q - A^*Q^{-q}A\right)^q - AQ^{-1}A^* > 0 \tag{5}$$

and

$$X^q \in \left(AQ^{-1}A^*,\ \left(Q - A^*Q^{-q}A\right)^q\right]. \tag{6}$$

Proof. Let X be a positive definite solution of Eq. (4), then $0 \leqslant A^*X^{-q}A < Q$. Hence, we obtain $X \leqslant Q$ and

$$X = Q - A^*X^{-q}A \leqslant Q - A^*Q^{-q}A. \tag{7}$$

Therefore

$$X^q \leqslant \left(Q - A^*Q^{-q}A\right)^q.$$

From $A^*X^{-q}A < Q$, it follows

$$\begin{aligned} Q^{-1/2}A^*X^{-q/2}X^{-q/2}AQ^{-1/2} &< I,\\ X^{-q/2}AQ^{-1}A^*X^{-q/2} &< I,\\ AQ^{-1}A^* &< X^q. \end{aligned}$$

Consequently

$$AQ^{-1}A^* < X^q \leqslant \left(Q - A^*Q^{-q}A\right)^q.$$

The theorem is proved. □

Theorem 2.3. *If Eq.* (4) *has a positive definite solution X, then*

$$X \in \left(\left(\frac{\mu}{\nu}\right)^{\frac{1-q}{q}}\left(AQ^{-1}A^*\right)^{\frac{1}{q}},\ Q - A^*Q^{-q}A\right], \tag{8}$$

where μ and ν are minimal and maximal eigenvalues of $AQ^{-1}A^*$, respectively.

Proof. Let X be a positive definite solution of Eq. (4), then by Theorem 2.2, statement (6), we have

$$AQ^{-1}A^* < X^q.$$

Let μ and ν are the minimal and maximal eigenvalues of $AQ^{-1}A^*$ respectively, then $\mu I \leqslant AQ^{-1}A^* \leqslant \nu I$. According to Theorem 2.1 in [10] for $q \leqslant 1$, we obtain

$$\left(AQ^{-1}A^*\right)^{\frac{1}{q}} < \left(\frac{\nu}{\mu}\right)^{\frac{1-q}{q}} X.$$

From $X > 0$, it follows that $0 \leqslant A^*X^{-q}A < Q$, $X \leqslant Q$ and

$$X = Q - A^*X^{-q}A \leqslant Q - A^*Q^{-q}A.$$

The theorem is proved. $\square$

The next theorem gives another upper bound of the positive definite solutions of Eq. (4).

Theorem 2.4. *If Eq.* (4) *has a positive definite solution X, then*

$$\sigma_m^2(Q^{-q/2}AQ^{-1/2}) \leqslant \frac{q^q}{(q+1)^{q+1}} \quad and \quad X \leqslant \hat{\alpha} Q,$$

where $\hat{\alpha}$ is a solution of the equation $\alpha^q(1-\alpha) = \sigma_m^2(Q^{-q/2}AQ^{-1/2})$ in $[\frac{q}{q+1}, 1]$.

Proof. We consider the sequence

$$\alpha_0 = 1, \quad \alpha_{s+1} = 1 - \frac{\sigma_m^2(Q^{-q/2}AQ^{-1/2})}{\alpha_s^q},\ s = 0, 1, 2, \ldots$$

Let X be a positive definite solution of Eq. (4). Then $X = Q - A^*X^{-q}A \leqslant Q$, i.e. $X \leqslant \alpha_0 Q$.

Assuming that $X \leqslant \alpha_s Q$, we have

$$\begin{aligned} X &= Q - A^*X^{-q}A \leqslant Q - A^*(\alpha_s Q)^{-q}A \\ &\leqslant Q^{1/2}\left(1 - \frac{\sigma_m^2(Q^{-q/2}AQ^{-1/2})}{\alpha_s^q}\right)Q^{1/2} = \alpha_{s+1}Q. \end{aligned}$$

Hence $X \leqslant \alpha_s Q$ for $s = 0, 1, 2, \ldots$ Obviously, the sequence $\{\alpha_s\}$ is monotonically decreasing. Hence $\{\alpha_s\}$ is convergent.

Let

$$\lim_{s\to\infty} \alpha_s = \hat{\alpha}, \quad \text{then } \hat{\alpha} = 1 - \frac{\sigma_m^2(Q^{-q/2}AQ^{-1/2})}{\hat{\alpha}^q},$$

i.e. $\hat{\alpha}$ is a solution of the equation $\alpha^q(1-\alpha) = \sigma_m^2(Q^{-q/2}AQ^{-1/2})$. Since

$$\max_{x\in[0,1]} f(x) = f\left(\frac{q}{q+1}\right) = \frac{q^q}{(q+1)^{q+1}},$$

where $f(x) = x^q(1-x)$ it follows that $\sigma_m^2(Q^{-q/2}AQ^{-1/2}) \leqslant \frac{q^q}{(q+1)^{q+1}}$.

In case

$$\sigma_m^2(Q^{-q/2}AQ^{-1/2}) \leqslant \frac{q^q}{(q+1)^{q+1}},$$

the equation

$$\alpha^q(1-\alpha) = \sigma_m^2(Q^{-q/2}AQ^{-1/2})$$

may have two solutions. One of these solutions is in the interval $[\frac{q}{q+1}, 1]$.

In order to prove that the limit $\hat{\alpha}$ of the sequence $\{\alpha_s\}$ is in $[\frac{q}{q+1}, 1]$ we assume that $\alpha_s > \frac{q}{q+1}$ (obviously $\alpha_0 = 1 > \frac{q}{q+1}$). Then

$$\begin{aligned}\alpha_{s+1} &= 1 - \frac{\sigma_m^2(Q^{-q/2}AQ^{-1/2})}{\alpha_s^q} \geqslant 1 - \frac{1}{\alpha_s^q}\frac{q^q}{(q+1)^{q+1}} \\ &> 1 - \left(\frac{q+1}{q}\right)^q \frac{q^q}{(q+1)^{q+1}} = \frac{q}{q+1}.\end{aligned}$$

Therefore, $\alpha_s > \frac{q}{q+1}$ for each $s = 0, 1, 2 \ldots$ Hence $\hat{\alpha} \geqslant \frac{q}{q+1}$. $\square$

Our next results are two iterative methods.

As first method we consider the iteration

$$X_0 = \gamma Q, \quad X_{s+1} = Q - A^* X_s^{-q} A, \; s = 0, 1, 2, \ldots \tag{9}$$

Theorem 2.5. *If Eq.* (4) *has a positive definite solution, then it has maximal one* X_L. *Moreover, the sequence* $\{X_s\}$ *in* (9) *for* $\gamma \in [\hat{\alpha}, 1]$ *is monotonically decreasing and converges to* X_L, *where* $\hat{\alpha}$ *is a solution in* $[\frac{q}{q+1}, 1]$ *of the equation*

$$\alpha^q(1-\alpha) = \sigma_m^2\left(Q^{-q/2}AQ^{-1/2}\right).$$

Proof. We consider the iterative method (9) with $\gamma \in [\hat{\alpha}, 1]$. According to Theorem 2.4 we have $X \leqslant \hat{\alpha}Q \leqslant \gamma Q = X_0$ for any positive definite solution X of Eq. (4).

We suppose that $X_s \geqslant X$, then

$$X_{s+1} = Q - A^* X_s^{-q} A \geqslant Q - A^* X^{-q} A = X.$$

Hence, for each s and any positive definite solution X, we have $X_s \geqslant X$. According to the definition of $\hat{\alpha}$ and the monotonicity of the function $x^q(1-x)$ on $[\frac{q}{q+1}, 1]$, for all $\gamma \in [\hat{\alpha}, 1]$ we have

$$\gamma^q(1-\gamma)I \leqslant Q^{-1/2}A^*Q^{-q}AQ^{-1/2}. \tag{10}$$

We compute

$$\begin{aligned} X_1 &= Q - A^*(\gamma Q)^{-q}A \\ &= Q^{1/2}\left(I - Q^{-1/2}A^*(\gamma Q)^{-q}AQ^{-1/2}\right)Q^{1/2}. \end{aligned}$$

Using inequality (10) we obtain

$$I - Q^{-1/2}A^*(\gamma Q)^{-q}AQ^{-1/2} \leqslant \gamma I.$$

Therefore $X_1 \leqslant X_0$. It is easy to prove by induction, that the sequence $\{X_s\}$ is monotonically decreasing. Hence, the sequence $\{X_s\}$ converges to the positive definite solution X_L of Eq. (4). Since $X_L \geqslant X$ for any positive definite solution X, it follows that X_L is the maximal solution. □

Theorem 2.6. *Let* $\sigma_m^2(B) < \sigma_1^2(B) \leqslant \frac{q^q}{(q+1)^{q+1}}$, *where* $q \in (0, 1]$ *and* $B = Q^{-q/2}AQ^{-1/2}$. *Let* $\hat{\alpha}$ *be a solution of the equation* $\alpha^q(1-\alpha) = \sigma_m^2(B)$ *in* $[\frac{q}{q+1}, 1]$ *and* β_1, β_2 *are solutions of* $\beta^q(1-\beta) = \sigma_1^2(B)$ *in* $[0, \frac{q}{q+1}]$ *and* $[\frac{q}{q+1}, 1]$, *respectively. Then*

(i) *if* $\gamma \in \left[\beta_1, \beta_2\right]$, *then the sequence* $\{X_s\}$ *in* (9) *is monotonically increasing and converges to a positive definite solution* $X_\gamma \in \left[\gamma Q, \hat{\alpha} Q\right]$ *of Eq.* (4),

(ii) *if* $\gamma \in \left[\hat{\alpha}, 1\right]$, *then the sequence* $\{X_s\}$ *in* (9) *is monotonically decreasing and converges to the maximal positive definite solution* $X_L \in \left[\beta_2 Q, \hat{\alpha} Q\right]$,

(iii) *if* $\gamma \in (\beta_2, \hat{\alpha})$ *and* $q\|A\|^2 < [\beta_2\lambda_m(Q)]^{q+1}$, *then the sequence* $\{X_s\}$ *in* (9) *converges to the unique solution* $X_L \in [\beta_2 Q, \hat{\alpha} Q]$.

Proof. We consider the function $f(x) = x^q(1-x)$, $x \in [0, 1]$. It is monotonically increasing on $[0, \frac{q}{q+1}]$ and monotonically decreasing on $[\frac{q}{q+1}, 1]$, and

$$\max_{x \in [0,1]} f(x) = f\left(\frac{q}{q+1}\right) = \frac{q^q}{(q+1)^{q+1}}.$$

Since $\sigma_1^2(B) \leqslant \frac{q^q}{(q+1)^{q+1}}$, then for each γ_1 and γ_2, such that $\beta_1 \leqslant \gamma_2 \leqslant \beta_2 < \hat{\alpha} \leqslant \gamma_1 \leqslant 1$ the inequalities

$$\gamma_1^q\,(1-\gamma_1)\,I \leqslant Q^{-1/2}A^*Q^{-q}AQ^{-1/2} \leqslant \gamma_2^q\,(1-\gamma_2)\,I \tag{11}$$

are satisfied.

(i) Let $\gamma \in \left[\beta_1, \beta_2\right]$. We will prove, that the matrix sequence $\{X_s\}$ in (9) is monotonically increasing and bounded above.

We compute

$$X_1 = Q - A^* (\gamma Q)^{-q} A = Q^{1/2} \left(I - Q^{-1/2} A^* (\gamma Q)^{-q} A Q^{-1/2} \right) Q^{1/2}.$$

According to the second inequality in (11), we get

$$I - Q^{-1/2} A^* (\gamma Q)^{-q} A Q^{-1/2} \geqslant \gamma I.$$

Hence $X_1 \geqslant \gamma Q = X_0$. Assuming that $X_s \geqslant X_{s-1}$, for X_{s+1} we have

$$X_{s+1} = Q - A^* X_s^{-q} A \geqslant Q - A^* X_{s-1}^{-q} A = X_s.$$

Therefore $X_{s+1} \geqslant X_s$ for $s = 0, 1, 2 \ldots,$ i.e. the sequence $\{X_s\}$ is monotonically increasing.

Obviously $X_0 = \gamma Q \leqslant \hat{\alpha} Q$. We suppose, that $X_s \leqslant \hat{\alpha} Q$. From the first inequality of (11), we obtain

$$X_{s+1} = Q - A^* X_s^{-q} A \leqslant Q - A^* \left(\hat{\alpha} Q\right)^{-q} A \leqslant \hat{\alpha} Q.$$

Hence, the matrix sequence $\{X_s\}$ converges to a positive definite solution X_γ of Eq. (4). Since $X_s \in \left[\gamma Q, \hat{\alpha} Q\right]$ for $s = 1, 2, \ldots,$ then $X_\gamma \in \left[\gamma Q, \hat{\alpha} Q\right]$.

(ii) Let $\gamma \in \left[\hat{\alpha}, 1\right]$. By analogy of the previous case, we prove that the sequence $\{X_s\}$ is monotonically decreasing and bounded below by $\beta_2 Q$. Hence $\{X_s\}$ is convergent. From Theorem 2.5 it follows that X_s converges to $X_L \in \left[\beta_2 Q, \hat{\alpha} Q\right]$.

(iii) We consider the sequence $\{X_s\}$ in (9) for $\gamma \in (\beta_2, \hat{\alpha})$, i.e. $X_0 \in (\beta_2 Q, \hat{\alpha} Q)$. We suppose that $X_s \in (\beta_2 Q, \hat{\alpha} Q)$. Then for X_{s+1}, we have

$$X_{s+1} = Q - A^* X_s^{-q} A < Q - A^* \left(\hat{\alpha} Q\right)^{-q} A \leqslant \hat{\alpha} Q,$$
$$X_{s+1} = Q - A^* X_s^{-q} A > Q - A^* (\beta_2 Q)^{-q} A \geqslant \beta_2 Q.$$

Hence, $X_s \in (\beta_2 Q, \hat{\alpha} Q)$ for $s = 0, 1, \ldots$

We consider $\|X_{s+1} - X_s\|$. According to Theorem X.3.8 in [2], we obtain

$$\begin{aligned}
\|X_{s+1} - X_s\| &= \left\| A^* X_{s-1}^{-q} (X_s^q - X_{s-1}^q) X_s^{-q} A \right\| \\
&\leqslant \left(\frac{\|A\|}{[\beta_2 \lambda_m(Q)]^q} \right)^2 \left\| X_s^q - X_{s-1}^q \right\| \\
&\leqslant \left(\frac{\|A\|}{[\beta_2 \lambda_m(Q)]^q} \right)^2 \frac{q}{[\beta_2 \lambda_m(Q)]^{1-q}} \|X_s - X_{s-1}\| \\
&= \frac{q \|A\|^2}{[\beta_2 \lambda_m(Q)]^{q+1}} \|X_s - X_{s-1}\|.
\end{aligned}$$

Since $q\|A\|^2 < \beta_2^{q+1}$, it follows that $\{X_s\}$ is a Cauchy sequence in the Banach space $[\beta_2 Q, \hat{\alpha} Q]$. Hence this sequence has a limit X_γ in $[\beta_2 Q, \hat{\alpha} Q]$ and X_γ is a unique solution of Eq. (4) in $[\beta_2 Q, \hat{\alpha} Q]$. According to Theorem 2.5, Eq. (4) has maximal positive definite solution X_L and $X_L \leqslant \hat{\alpha} Q$. Therefore $X_\gamma \equiv X_L$. □

We consider the second iterative method

$$Y_0 = \eta I, \quad Y_{s+1} = \left[A(Q - Y_s)^{-1}A^*\right]^{\theta}, \ s = 0, 1, 2, \ldots \left(\theta = \frac{1}{q} \geqslant 1\right). \tag{12}$$

Theorem 2.7. *Let* $0 < \sigma_m^2(A) < \sigma_1^2(A) \leqslant q^q \left(\frac{\lambda_m(Q)}{q+1}\right)^{q+1}$. *Let* $\tilde{\alpha}$ *be a solution of the equation* $x^q(1-x) = \frac{\sigma_m^2(A)}{\lambda_1^{q+1}(Q)}$ *in* $(0, \frac{q}{q+1}]$, *and* $\tilde{\beta}_1, \tilde{\beta}_2$ *are solutions of the equation* $x^q(1-x) = \frac{\sigma_1^2(A)}{\lambda_m^{q+1}(Q)}$ *in* $(0, \frac{q}{q+1}]$ *and* $[\frac{q}{q+1}, 1)$, *respectively. Then Eq.* (4) *has a positive definite solution in* $[\tilde{\alpha}\lambda_1(Q)I, \tilde{\beta}_1\lambda_m(Q)I]$. *Moreover*,

(i) *if* $\frac{\sigma_1^{2k}(A)}{\lambda_1^{kq-1}(Q)\lambda_m^{k+1}(Q)} < q\tilde{\alpha}^{kq-1}(1-\tilde{\beta}_1)^{k+1}$, *where* $k \in \mathbb{N}$ *such that* $\frac{1}{q} \in (k-1, k]$, *then Eq.* (4) *has a unique positive definite solution* $\overline{Y} \in [\tilde{\alpha}\lambda_1(Q)I, \ \tilde{\beta}_1\lambda_m(Q)I]$,

(ii) *if for any* α *and* β, *such that* $0 < \alpha \leqslant \tilde{\alpha} < \tilde{\beta}_1 \leqslant \beta \leqslant \tilde{\beta}_2 < 1$ *the inequality* $\frac{\sigma_1^{2k}(A)}{\lambda_1^{kq-1}(Q)\lambda_m^{k+1}(Q)} < q\alpha^{kq-1}(1-\beta)^{k+1}$ *is satisfied, then the unique positive definite solution* $\overline{Y} \in [\tilde{\alpha}\lambda_1(Q)I, \tilde{\beta}_1\lambda_m(Q)I]$ *of Eq.* (4) *is a unique solution in* $\left[\alpha\lambda_1(Q)I, \beta\lambda_m(Q)I\right]$, *which is the limit of the matrix sequence* $\{Y_s\}$ *defined by* (12) *for any* $\eta \in [\alpha\lambda_1(Q), \beta\lambda_m(Q)]$.

Proof. The function $f(x) = x^q(1-x)$, $x \in [0, 1]$ is monotonically increasing in $[0, \frac{q}{q+1}]$ and monotonically decreasing in $[\frac{q}{q+1}, 1]$, and $\max_{x\in[0,1]} f(x) = f(\frac{q}{q+1}) = \frac{q^q}{(q+1)^{q+1}}$.

Since $\sigma_1^2(A) \leqslant q^q(\frac{\lambda_m(Q)}{q+1})^{q+1}$, it follows that for any α and β, such that $0 < \alpha \leqslant \tilde{\alpha} < \tilde{\beta}_1 \leqslant \beta \leq \tilde{\beta}_2 < 1$ the inequalities

$$\alpha^q\,(1-\alpha)\,\lambda_1^{q+1}(Q)I \leqslant AA^* \leqslant \beta^q\,(1-\beta)\,\lambda_m^{q+1}(Q)I \tag{13}$$

are satisfied.

We consider the map

$$\mathcal{G}(Y) \equiv \left[A(Q-Y)^{-1}A^*\right]^{\theta}.$$

Let $Y \in \left[\alpha\lambda_1(Q)I, \beta\lambda_m(Q)I\right]$. Then

$$\frac{1}{(1-\alpha)\lambda_1(Q)}I \leqslant (Q-Y)^{-1} \leqslant \frac{1}{(1-\beta)\lambda_m(Q)}I. \tag{14}$$

(13) and (14) imply

$$A(Q-Y)^{-1}A^* \leqslant \frac{1}{(1-\beta)\lambda_m(Q)}AA^* \leqslant \left[\beta\lambda_m(Q)\right]^q I$$

and

$$A(Q-Y)^{-1}A^* \geqslant \frac{1}{(1-\alpha)\lambda_1(Q)}AA^* \geqslant [\alpha\lambda_1(Q)]^q\, I.$$

Hence $\mathscr{G}(Y) \in \left[\alpha\lambda_1(Q)I, \beta\lambda_m(Q)I\right]$. Since $\mathscr{G}$ is continuous on $\left[\alpha\lambda_1(Q)I, \beta\lambda_m(Q)I\right]$, then by Schauder's fixed point theorem there exists a matrix Y in $\left[\alpha\lambda_1(Q)I, \beta\lambda_m(Q)I\right]$, such that $\mathscr{G}(Y) = Y$. Since A is nonsingular matrix, it follows that Y satisfies Eq. (4). By definitions of the α and β, it follows that they can be equal to $\tilde{\alpha}$ and $\tilde{\beta}_1$, respectively.

Therefore, Eq. (4) has a positive definite solution in $\left[\tilde{\alpha}\lambda_1(Q)I, \tilde{\beta}_1\lambda_m(Q)I\right]$.

We suppose that Y and $\widetilde{Y}$ are two different matrices in $\left[\alpha\lambda_1(Q)I, \beta\lambda_m(Q)I\right]$ and $P = A(Q-Y)^{-1}A^*$, $\widetilde{P} = A(Q-\widetilde{Y})^{-1}A^*$, then

$$\begin{aligned}\mathscr{G}(Y) - \mathscr{G}(\widetilde{Y}) &= \left[A(Q-Y)^{-1}A^*\right]^\theta - \left[A(Q-\widetilde{Y})^{-1}A^*\right]^\theta \\ &= P^\theta - \widetilde{P}^\theta = \left(P^k\right)^{\frac{\theta}{k}} - \left(\widetilde{P}^k\right)^{\frac{\theta}{k}},\end{aligned}$$

where $k \in \mathbb{N}$ and $\theta \in (k-1, k]$. Since $P^k \geqslant [\alpha\lambda_1(Q)]^{kq}I$, $\widetilde{P}^k \geqslant [\alpha\lambda_1(Q)]^{kq}I$ and $\frac{\theta}{k} \leqslant 1$, then according to Theorem X.3.8 in [2], we have

$$\begin{aligned}\left\|\mathscr{G}(Y) - \mathscr{G}(\widetilde{Y})\right\| &= \left\|(P^k)^{\frac{\theta}{k}} - (\widetilde{P}^k)^{\frac{\theta}{k}}\right\| \leqslant \frac{\theta}{k}\,[\alpha\lambda_1(Q)]^{kq\left(\frac{\theta}{k}-1\right)}\left\|P^k - \widetilde{P}^k\right\| \\ &= \frac{\theta}{k}\,[\alpha\lambda_1(Q)]^{1-kq}\left\|\sum_{i=0}^{k-1} P^{k-i-1}(P-\widetilde{P})\widetilde{P}^i\right\| \\ &\leqslant \theta\,[\alpha\lambda_1(Q)]^{1-kq}\,\frac{\sigma_1^{2(k-1)}(A)}{(1-\beta)^{k-1}\lambda_m^{k-1}(Q)}\|P-\widetilde{P}\|.\end{aligned}$$

By the expressions of P and $\widetilde{P}$, we get

$$\begin{aligned}\|P-\widetilde{P}\| &= \|A(Q-Y)^{-1}(\widetilde{Y}-Y)(Q-\widetilde{Y})^{-1}A^*\| \\ &\leqslant \|A(Q-Y)^{-1}\|\|(Q-\widetilde{Y})^{-1}A^*\|\|\widetilde{Y}-Y\| \\ &\leqslant \frac{\sigma_1^2(A)}{(1-\beta)^2\lambda_m^2(Q)}\|\widetilde{Y}-Y\|.\end{aligned}$$

Consequently,

$$\left\|\mathscr{G}(Y) - \mathscr{G}(\widetilde{Y})\right\| \leqslant \frac{1}{q\alpha^{kq-1}(1-\beta)^{k+1}}\,\frac{\sigma_1^{2k}(A)}{\lambda_1^{kq-1}(Q)\lambda_m^{k+1}(Q)}\|\widetilde{Y}-Y\|.$$

For $\alpha = \tilde{\alpha}$ and $\beta = \tilde{\beta}_1$ in (i), we have

$$\frac{1}{q\alpha^{kq-1}(1-\beta)^{k+1}}\,\frac{\sigma_1^{2k}(A)}{\lambda_1^{kq-1}(Q)\lambda_m^{k+1}(Q)} < 1.$$

Hence $\mathscr{G}$ is contractive map on $[\tilde{\alpha}\lambda_1(Q)I,\ \tilde{\beta}_1\lambda_m(Q)I]$. According to Banach's fixed point theorem there exists a unique matrix $\overline{Y} \in [\tilde{\alpha}\lambda_1(Q)I,\ \tilde{\beta}_1\lambda_m(Q)I]$, such that $\mathscr{G}(\overline{Y}) = \overline{Y}$.

The above result is true for any α and β for which the conditions in (ii) are satisfied. Since $[\tilde{\alpha}\lambda_1(Q)I, \tilde{\beta}_1\lambda_m(Q)I] \subset [\alpha\lambda_1(Q)I, \beta\lambda_m(Q)I]$, then the unique solution in $[\alpha\lambda_1(Q)I, \beta\lambda_m(Q)I]$ is the $\bar{Y} \in [\tilde{\alpha}\lambda_1(Q)I,\ \tilde{\beta}_1\lambda_m(Q)I]$.

We consider the method (12) with $\eta \in [\alpha\lambda_1(Q), \beta\lambda_m(Q)]$, i.e.

$$Y_0 \in [\alpha\lambda_1(Q)I, \beta\lambda_m(Q)I].$$

Since $\mathscr{G}([\alpha\lambda_1(Q)I, \beta\lambda_m(Q)I]) \subset [\alpha\lambda_1(Q)I, \beta\lambda_m(Q)I]$, then $Y_s \in [\alpha\lambda_1(Q)I, \beta\lambda_m(Q)I]$ for each s. Similar of the above manner, we get

$$\|Y_s - \overline{Y}\| \leqslant \frac{1}{q\alpha^{kq-1}(1-\beta)^{k+1}} \frac{\sigma_1^{2k}(A)}{\lambda_1^{kq-1}(Q)\lambda_m^{k+1}(Q)} \|Y_{s-1} - \overline{Y}\|.$$

Since

$$\frac{1}{q\alpha^{kq-1}(1-\beta)^{k+1}} \frac{\sigma_1^{2k}(A)}{\lambda_1^{kq-1}(Q)\lambda_m^{k+1}(Q)} < 1,$$

it follows that $\overline{Y}$ is the limit of the sequence $\{Y_s\}$. $\square$

Remark 2.1. When $q = \frac{1}{k}$, $k = 1, 2, \ldots,$ from the condition $\sigma_1^2(A) \leqslant q^q \left(\frac{\lambda_m(Q)}{q+1}\right)^{q+1}$ it follows

$$\frac{\sigma_1^{2k}(A)}{\lambda_1^{kq-1}(Q)\lambda_m^{k+1}(Q)} < q\tilde{\alpha}^{kq-1}(1-\tilde{\beta}_1)^{k+1}.$$

3. On the equation $X - A^*X^{-q}A = Q$

In this section we consider the equation

$$X - A^*X^{-q}A = Q. \tag{15}$$

Eq. (15) can be got from (3) by $\mathscr{F}(X) = -X^{-q}$. The next theorem is proved by Ran and Reurings in [18].

Theorem 3.1 [18, Lemma 2.2]. *Let $\mathscr{F} : \mathscr{P}(m) \to -\mathscr{P}(m)$ be continuous on $\{X \in \mathscr{P}(m) | X \geqslant Q\}$.*

(i) *If (3) has a positive definite solution $\bar{X}$, then $\bar{X} \geqslant Q$.*

(ii) *If there exists a $B \geqslant Q$ such that*

$$Q - B \leqslant A^*\mathscr{F}(X)A \leqslant 0 \tag{16}$$

for all $X \in [Q, B]$, then (3) has a solution in $[Q, B]$. Moreover, if (16) is satisfied for every $X \geqslant Q$, then all solutions of (3) are in $[Q, B]$.

Corollary 3.2. *Eq.* (15) *has a positive definite solution X and all the positive definite solutions are in* $\left[Q, Q + A^*Q^{-q}A\right]$.

Proof. We consider Theorem 3.1 with $\mathscr{F}(X) = -X^{-q}$ and $B = Q + A^*X^{-q}A$. From (ii) we obtain the request result. □

The next theorem is similar to Liu's and Gao's Theorem 2.5 in [16] for the equation $X^s - A^{\mathrm{T}}X^{-t}A = I,\ s, t \in \mathbb{N}$.

Theorem 3.3. *Every positive definite solution X of Eq.* (15) *is in* $\left[\alpha I, \beta I\right]$, *where the pair* (α, β) *is a solution of the system*

$$\left| \begin{aligned} &\alpha = \lambda_m(Q) + \sigma_m^2(A)\beta^{-q}, \\ &\beta = \lambda_1(Q) + \sigma_1^2(A)\alpha^{-q}. \end{aligned} \right. \tag{17}$$

Proof. We define the sequences $\{\alpha_s\}$ and $\{\beta_s\}$

$$\alpha_0 = \lambda_m(Q), \quad \beta_0 = \lambda_1(Q) + \frac{\sigma_1^2(A)}{\lambda_m^q(Q)},$$

$$\alpha_s = \lambda_m(Q) + \sigma_m^2(A)\beta_{s-1}^{-q}, \tag{18}$$

$$\beta_s = \lambda_1(Q) + \sigma_1^2(A)\alpha_s^{-q}, \quad s = 1, 2, \ldots \tag{19}$$

We will prove that the sequences $\{\alpha_s\}$ and $\{\beta_s\}$ are monotonically increasing and monotonically decreasing, respectively. Moreover, for any positive definite solution X, $X \in \left[\alpha_s I, \beta_s I\right]$, $s = 0, 1, 2, \ldots$

By definition $0 < \alpha_0 < \beta_0$. Hence

$$\alpha_1 = \alpha_0 + \sigma_m^2(A)\beta_0^{-q} \geqslant \alpha_0$$

and

$$\beta_1 = \lambda_1(Q) + \sigma_1^2(A)\alpha_1^{-q} \leqslant \lambda_1(Q) + \sigma_1^2(A)\alpha_0^{-q} = \beta_0.$$

We suppose, that $\alpha_k \geqslant \alpha_{k-1}$ and $\beta_k \leqslant \beta_{k-1}$. Then

$$\alpha_{k+1} = \lambda_m(Q) + \sigma_m^2(A)\beta_k^{-q} \geqslant \lambda_m(Q) + \sigma_m^2(A)\beta_{k-1}^{-q} = \alpha_k$$

and

$$\beta_{k+1} = \lambda_1(Q) + \sigma_1^2(A)\alpha_{k+1}^{-q} \leqslant \lambda_1(Q) + \sigma_1^2(A)\alpha_k^{-q} = \beta_k.$$

Hence, for each s we have $\alpha_{s+1} \geqslant \alpha_s$ and $\beta_{s+1} \leqslant \beta_s$.

We will show that $X \in \left[\alpha_s I, \beta_s I\right]$ for any positive definite solution X and $s = 0, 1, \ldots$ According to Corollary 3.2 we have

$$Q \leqslant X \leqslant Q + A^*Q^{-q}A$$

for each positive definite solution X. Since $Q \geqslant \lambda_m(Q)I = \alpha_0 I$, then $X \geqslant \alpha_0 I$. On other hand

$$Q + A^* Q^{-q} A \leqslant Q + \frac{A^* A}{\lambda_m^q(Q)} \leqslant \lambda_1(Q) + \frac{\sigma_1^2(A)}{\lambda_m^q(Q)} = \beta_0 I$$

and thus $X \leqslant \beta_0 I$. Therefore $X \in [\alpha_0 I, \beta_0 I]$.

We suppose, that $X \in [\alpha_k I, \beta_k I]$. Then

$$\begin{aligned} X = Q + A^* X^{-q} A \ &\geqslant \ Q + \beta_k^{-q} A^* A \\ &\geqslant \lambda_m(Q) I + \sigma_m^2(A) \beta_k^{-q} I \ = \ \alpha_{k+1} I, \end{aligned}$$

and

$$\begin{aligned} X = Q + A^* X^{-q} A &\leqslant Q + \alpha_{k+1}^{-q} A^* A \\ &\leqslant \lambda_1(Q) I + \sigma_1^2(A) \alpha_{k+1}^{-q} I = \beta_{k+1} I. \end{aligned}$$

Hence $\alpha_s I \leqslant X \leqslant \beta_s I$ for all s.

Consequently, the sequences $\{\alpha_s\}$ and $\{\beta_s\}$ are convergent.

Let

$$\alpha = \lim_{s \to \infty} \alpha_s, \quad \beta = \lim_{s \to \infty} \beta_s.$$

Then $X \in [\alpha I, \beta I]$. Taking limits in (18) and (19) yields

$$\begin{aligned} \alpha &= \lambda_m(Q) + \sigma_m^2(A) \beta^{-q} \\ \beta &= \lambda_1(Q) + \sigma_1^2(A) \alpha^{-q}. \end{aligned}$$

Therefore α and β satisfy the system (17). $\square$

Theorem 3.4. *Every positive definite solution X of Eq. (15) is in $[\alpha' Q, \beta' Q]$, where the pair (α', β') is a solution of the system*

$$\left| \begin{aligned} \alpha' &= 1 + \sigma_m^2(B) \beta'^{-q}, \\ \beta' &= 1 + \sigma_1^2(B) \alpha'^{-q} \end{aligned} \right.$$

with $B = Q^{-q/2} A Q^{-1/2}$.

Proof. The proof is similar to the proof of Theorem 3.3, as we can consider the sequences

$$\begin{aligned} \alpha'_0 &= 1, \quad \beta'_0 = 1 + \sigma_1^2(B), \\ \alpha'_s &= 1 + \frac{\sigma_m^2(B)}{(\beta'_{s-1})^q}, \\ \beta'_s &= 1 + \frac{\sigma_1^2(B)}{\alpha_s'^{\,q}}, \quad s = 1, 2, \ldots \end{aligned}$$

$\square$

Theorem 3.5. *If $q\,\|A\|^2 < \alpha^{q+1}$, where α is from Theorem 3.3, then Eq. (15) has a unique positive definite solution.*

Proof. We suppose that X and Y are two different positive definite solutions of Eq. (15). Then by Theorem 3.3 it follows $X \geqslant \alpha I$, $Y \geqslant \alpha I$.

$$X - Y = A^*\left(X^{-q} - Y^{-q}\right)A = A^*X^{-q}\left(Y^q - X^q\right)Y^{-q}A.$$

According to Theorem X.3.8 in [2], we get

$$\begin{aligned}\|X-Y\| &\leqslant \|A\|^2\|X^{-q}\|\|Y^{-q}\|\|Y^q - X^q\| \\ &\leqslant \left(\frac{\|A\|}{\alpha^q}\right)^2 q\alpha^{q-1}\|Y-X\| = q\frac{\|A\|^2}{\alpha^{q+1}}\|Y-X\|.\end{aligned}$$

Since $q\|A\|^2 < \alpha^{q+1}$, it follows $X \equiv Y$. □

Corollary 3.6. *If $q\|A\|^2 < \lambda_m^{q+1}(Q)$, then Eq. (15) has a unique positive definite solution.*

Proof. We have $\lambda_m(Q) \leqslant \alpha$, where α is from Theorem 3.3. Hence

$$q\|A\|^2 < \lambda_m^{q+1}(Q) \leqslant \alpha^{q+1}.$$

Theorem 3.5 implies that Eq. (15) has a unique positive definite solution. □

To solve Eq. (15), we consider the iteration

$$X_0 = \gamma Q, \quad X_{s+1} = Q + A^*X_s^{-q}A, \; s = 0, 1, 2, \ldots \tag{20}$$

Theorem 3.7. *Let $\tilde{\eta} \geqslant 1$ be a solution of $x^q(x-1) = \sigma_m^2(Q^{-q/2}AQ^{-1/2})$. If the matrices A and Q, and number $\eta \geqslant 1$ satisfy the inequalities*

(i) $\eta^q(\eta-1)Q \leqslant A^*Q^{-q}A$,
(ii) $(\eta-1)\eta^{-q^2}Q \leqslant A^*\left(\eta^q Q + A^*Q^{-q}A\right)^{-q}A$,
(iii) $q\,\|A\|^2 < \eta^{q+1}\lambda_m^{q+1}(Q)$,

then the sequence $\{X_s\}$ defined in (20) for all $\gamma \in [\eta, \tilde{\eta}]$ converges to a unique positive definite solution $X \geqslant \gamma Q$ of Eq. (15).

Proof. We consider the matrix sequence $\{X_s\}$ defined in (20). First, we will prove the convergence of $\{X_s\}$ by $\gamma = \eta$, i.e. $X_0 = \eta Q$. We have

$$X_1 = Q + \eta^{-q}A^*Q^{-q}A.$$

By condition (i) we obtain

$$\begin{aligned}&\eta^q(\eta-1)Q \leqslant A^*Q^{-q}A \\ &\quad X_0 = \eta Q \leqslant Q + \eta^{-q}A^*Q^{-q}A = X_1.\end{aligned}$$

For X_2, we find

$$X_2 = Q + A^* X_1^{-q} A = Q + A^* \left(Q + \eta^{-q} A^* Q^{-q} A\right)^{-q} A$$
$$= Q + \eta^{q^2} A^* \left(\eta^q Q + A^* Q^{-q} A\right)^{-q} A.$$

Using condition (ii), we get $X_0 \leqslant X_2$. Since $X_0 \leqslant X_1$, then $X_2 \leqslant X_1$.

Consequently, $X_0 \leqslant X_2 \leqslant X_1$.

By induction, for the elements of the sequence $\{X_s\}$ we prove that for any positive integers p and r the inequalities

$$X_0 \leqslant X_{2p} \leqslant X_{2p+2} \leqslant X_{2r+3} \leqslant X_{2r+1} \leqslant X_1 \tag{21}$$

are satisfied.

Hence the subsequences $\left\{X_{2p}\right\}$ and $\{X_{2r+1}\}$ are convergent. We will prove that these sequences have a common limit. For that purpose we consider $\|X_{2k+1} - X_{2k}\|$. By analogy to the proof of Theorem 3.5, we get

$$\begin{aligned}\|X_{2k+1} - X_{2k}\| &= \left\|A^* X_{2k-1}^{-q} \left(X_{2k}^q - X_{2k-1}^q\right) X_{2k}^{-q} A\right\| \\ &\leqslant \|A\|^2 \|X_0^{-1}\|^{2q} \left\|X_{2k}^q - X_{2k-1}^q\right\| \\ &= \frac{\|A\|^2}{\eta^{2q}\lambda_m^{2q}(Q)} \left\|X_{2k}^q - X_{2k-1}^q\right\| \\ &\leqslant \frac{q\|A\|^2}{\eta^{q+1}\lambda_m^{q+1}(Q)} \|X_{2k} - X_{2k-1}\|,\end{aligned}$$

where $\frac{q\|A\|^2}{\eta^{q+1}\lambda_m^{q+1}(Q)} < 1$. Hence the two sequences $\left\{X_{2p}\right\}$ and $\left\{X_{2q+1}\right\}$ have the same limit which is a solution of Eq. (15).

Now, we will prove the convergence of $\{X_s\}$ for any $\gamma \in [\eta, \tilde{\eta}]$. We denote the elements of the sequence $\{X_s\}$ for arbitrary γ with X'_s. Since

$$\tilde{\eta}^q(\tilde{\eta} - 1) = \sigma_m^2(Q^{-q/2} A Q^{-1/2}),$$

it follows that $\tilde{\eta}^q(\tilde{\eta} - 1)Q \leqslant A^* Q^{-q} A$. For $\gamma \in [\eta, \tilde{\eta}]$ we have

$$\gamma^q(\gamma - 1)Q \leqslant \tilde{\eta}^q(\tilde{\eta} - 1)Q \leqslant A^* Q^{-q} A.$$

Hence $X'_0 = \gamma Q \leqslant X'_1$ and $X_0 = \eta Q \leqslant X'_0 \leqslant X'_1$. Moreover,

$$X_1 = Q + A^* X_0^{-q} A \geqslant Q + A^* X_0'^{-q} A = X'_1$$

and thus $X_2 \leqslant X'_2$. From $X'_0 \leqslant X'_1$, it follows $X'_2 \leqslant X'_1$.

Therefore

$$X_0 \leqslant X'_0 \leqslant X'_1 \leqslant X_1 \quad \text{and} \quad X_2 \leqslant X'_2 \leqslant X'_1 \leqslant X_1.$$

By induction, we get

$$X_{2p} \leqslant X'_{2p} \leqslant X'_{2p+1} \leqslant X_{2p+1},$$

$$X_{2p+2} \leqslant X'_{2p+2} \leqslant X'_{2p+1} \leqslant X_{2p+1}, \quad p = 0, 1, \ldots$$

From the convergence of $\{X_s\}$ with $X_0 = \eta Q$, it follows the convergence of $\left\{X'_s\right\}$. Hence, the sequence $\{X_s\}$ is convergent for any $\gamma \in [\eta, \tilde{\eta}]$.

The uniqueness of the solution $X \geqslant \eta Q$ is obviously. We suppose that Eq. (15) has two different solutions $X \geqslant \eta Q$ and $Y \geqslant \eta Q$, then we get

$$\|X - Y\| \leqslant \frac{q\|A\|^2}{\eta^{q+1}\lambda_m^{q+1}(Q)} \|X - Y\| < \|X - Y\|,$$

which is a contradiction. □

Corollary 3.8. *If $q\|A\|^2 < \lambda_m^{q+1}(Q)$, then the iterative method (20) with $X_0 = Q$ converges to a unique positive definite solution of Eq. (15).*

Proof. According to Corollary 3.6 Eq. (15) has a unique positive definite solution. Obviously, for $\gamma = 1$ the three conditions (i), (ii) and (iii) of Theorem 3.7 are satisfied. Hence, the method (20) converges to this solution. □

Theorem 3.9. *If there are numbers $\eta \geqslant 1$ and $\xi \geqslant \eta$, for which the matrices A and Q satisfy the inequalities*

(i) $\xi^q(\eta - 1)Q \leqslant A^*Q^{-q}A \leqslant \eta^q(\xi - 1)Q$,
(ii) $q\,\|A\|^2 < \eta^{q+1}\lambda_m^{q+1}(Q)$,

then Eq. (15) has a unique positive definite solution X in $\left[\eta Q, \xi Q\right]$. Moreover, the iteration (20) converges to X for all $\gamma \in [\eta, \xi]$.

Proof. Let $\gamma \in [\eta, \xi]$, i.e. $X_0 \in [\eta Q, \xi Q]$. From $\xi^q(\eta - 1)Q \leqslant A^*Q^{-q}A$, we have $(\eta - 1)Q \leqslant \xi^{-q}A^*Q^{-q}A$. Hence

$$X_1 = Q + \gamma^{-q}A^*Q^{-q}A \geqslant Q + \xi^{-q}A^*Q^{-q}A \geqslant \eta Q.$$

From the second inequality in (i), we obtain $\eta^{-q}A^*Q^{-q}A \leqslant (\xi - 1)Q$ and

$$X_1 = Q + \gamma^{-q}A^*Q^{-q}A \leqslant Q + \eta^{-q}A^*Q^{-q}A \leqslant \xi Q.$$

Hence $X_1 \in [\eta Q, \xi Q]$.

By induction we have $X_s \in [\eta Q, \xi Q]$ for each $s = 0, 1, \ldots$

We consider

$$\begin{aligned}\|X_{s+1} - X_s\| &= \left\|A^*X_s^{-q}(X_{s-1}^q - X_s^q)X_{s-1}^{-q}A\right\| \\ &\leqslant \frac{\|A\|^2}{\eta^{2q}\lambda_m^{2q}(Q)}\left\|X_{s-1}^q - X_s^q\right\| \leqslant \frac{q\|A\|^2}{\eta^{q+1}\lambda_m^{q+1}(Q)}\|X_{s-1} - X_s\|.\end{aligned}$$

By condition (ii), we have $\frac{\|A\|^2}{q\eta^{q+1}\lambda_m^{q+1}(Q)} < 1$, hence X_s is convergent as a Cauchy sequence and its limit $X \in [\eta Q, \xi Q]$ is a solution of Eq. (15). $\square$

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